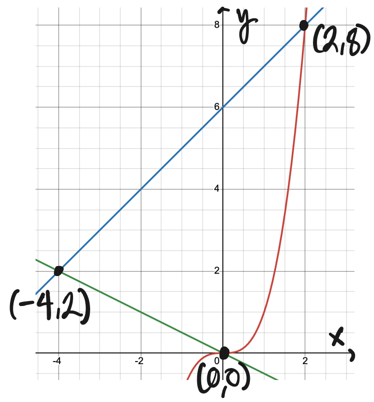
① Determine the area of the region given below bounded by $y = x^3$, $y - x = 6$, and $2y = -x$.

To determine the intersections, use the graph to identify the coordinates.



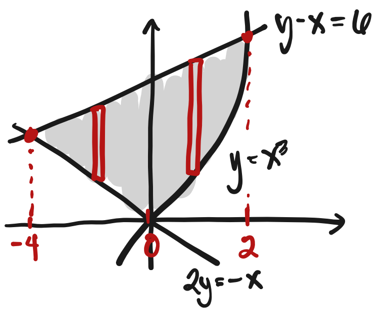
① Between $y = x^3$ and $y - x = 6$, $y = x + 6$:
If $x = 2$: $y = (2)^3 = 8$ and $y = 2 + 6 = 8$. Intersection @ $(2,8)$

② Between $y = x + 6$ and $2y = -x$, $y = -\frac{1}{2}x$:
If $x = -4$: $y = -4 + 6 = 2$ and $y = -\frac{1}{2}(-4) = 2$. Intersection @ $(-4,2)$

③ Between $y = x^3$ and $y = -\frac{1}{2}x$:
If $x = 0$: $y = (0)^3 = 0$ and $y = -\frac{1}{2}(0) = 0$. Intersection @ $(0,0)$

For the area integral: this can be done in two ways:

Option A: Integrate with respect to x, i.e. slice perpendicular to the x-axis. (Preferred)



thickness: dx
bounds: $x \in [-4, 2]$
height: y_{high}: $y - x = 6$. So, $y_{high} = x + 6$;
y_{low}: On $x \in [-4, 0]$: $2y = -x$; So, $y_{low} = -\frac{1}{2}x$;
On $x \in [0, 2]$: $y = x^3$; So, $y_{low} = x^3$;

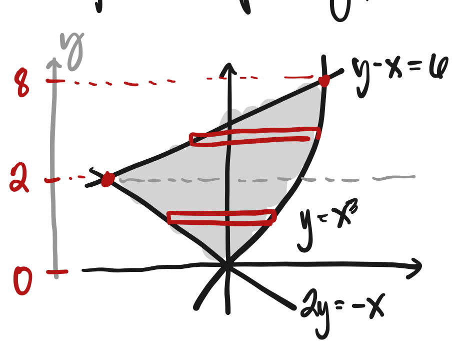
$$\boxed{A = \int_{-4}^{0} (x+6) - \left(-\tfrac{1}{2}x\right) dx + \int_{0}^{2} (x+6) - (x^3)\, dx}$$

$$A_1 = \int_{-4}^{0} \tfrac{3}{2}x + 6\, dx = \left[\tfrac{3}{2}\left(\tfrac{1}{2}\right)x^2 + 6x\right]_{-4}^{0} = \tfrac{3}{4}\left[(0)^2 - (-4)^2\right] + 6\left[(0) - (-4)\right] = 12;$$

$$A_2 = \int_{0}^{2} -x^3 + x + 6\, dx = \left[-\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2 + 6x\right]_{0}^{2} = -\tfrac{1}{4}\left[2^4 - 0\right] + \tfrac{1}{2}\left[2^2 - 0\right] + 6\left[2 - 0\right] = 10;$$

$$A = A_1 + A_2 = 12 + 10 = \boxed{22}$$

Option B. Integrate with respect to y, i.e. slice perpendicular to the y-axis.



thickness: dy
bounds: $y \in [0, 8]$
height: x_{right}: $y = x^3$, $x = y^{\frac{1}{3}}$; So, $x_{right} = y^{\frac{1}{3}}$;
x_{left}: On $y \in [0, 2]$: $2y = -x$, $x = -2y$; $x_{left} = -2y$.
On $y \in [2, 8]$: $y - x = 6$, $x = y - 6$; $x_{left} = y - 6$

$$\boxed{A = \int_{0}^{2} y^{\frac{1}{3}} - (-2y)\, dy + \int_{2}^{8} y^{\frac{1}{3}} - (y-6)\, dy}$$

$$A_1 = \int_0^2 y^{\frac{1}{3}} + 2y \, dy = \left[\frac{3}{4} y^{\frac{4}{3}} + \cancel{2}\left(\tfrac{1}{\cancel{2}}\right) y^2 \right]_0^2 = \frac{3}{4}\left[2^{\frac{4}{3}} - 0^{\frac{4}{3}}\right] + \left[2^2 - 0\right] = \frac{3}{4}\left(2^{\frac{4}{3}}\right) + 4 \, ;$$

$$A_2 = \int_2^8 y^{\frac{1}{3}} - y + 6 \, dy = \left[\frac{3}{4} y^{\frac{4}{3}} - \frac{1}{2} y^2 + 6y \right]_2^8 = \frac{3}{4}\left[8^{\frac{4}{3}} - 2^{\frac{4}{3}}\right] - \frac{1}{2}\left[8^2 - 2^2\right] + 6(8-2)$$

$$= \frac{3}{4}(16) - \frac{3}{4}\left(2^{\frac{4}{3}}\right) - \frac{1}{2}(60) + 36 = 12 - \frac{3}{4}\left(2^{\frac{4}{3}}\right) - 30 + 36 = 18 - \frac{3}{4}\left(2^{\frac{4}{3}}\right) \, ;$$

$$A = A_1 + A_2 = \cancel{\frac{3}{4}\left(2^{\frac{4}{3}}\right)} + 4 + 18 - \cancel{\frac{3}{4}\left(2^{\frac{4}{3}}\right)} = \boxed{22}$$

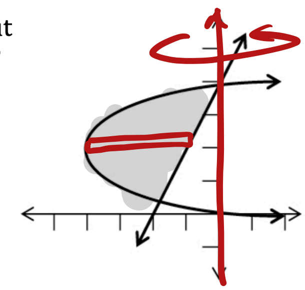
2. Use the diagrams below to answer the following questions:

a) Suppose the bounded region in the graph to the right is rotated about the *y-axis*. Which integration method will require only one integral? (Circle one)

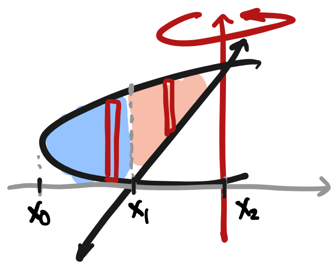
slice perpendicular to the axis of rotation

Washer Method **Shell Method**

Draw in an appropriate rectangular slice in the diagram for the method you chose.



If we use the Shell Method, we slice parallel to the axis of rotation.



The lower functions for the height of the cylindrical shells are different for $x \in [x_0, x_1]$ and $x \in [x_1, x_2]$.
So, we need 2 integrals for the Shell Method.

Sidenote: We also prefer the washer method here since if we use the Shell Method: The upper and lower functions for the height of the shells on $x \in [x_0, x_1]$ are given by the same curve.

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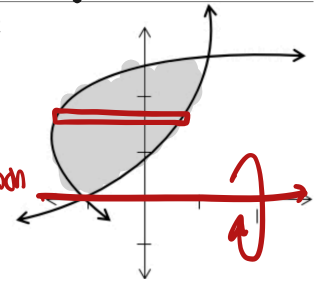
b) Suppose the bounded region in the graph to the right is rotated about the *x-axis*. Which integration method will require only one integral? (Circle one)

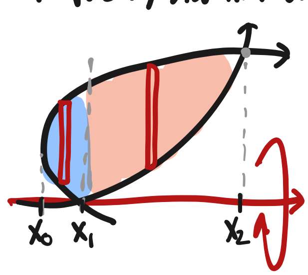
Washer Method **Shell Method**

slice parallel to the axis of rotation

Draw in an appropriate rectangular slice in the diagram for the method you chose.



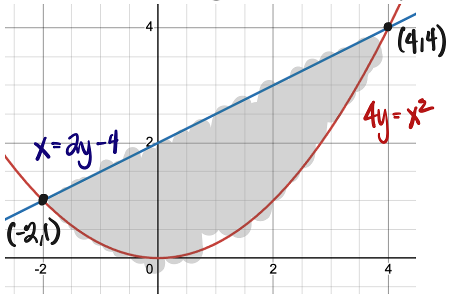
If we use the Washer/Disk Method, we slice perpendicular to the axis of rotation.



The functions for the inner radius of the washers are different for $x \in [x_0, x_1]$ and for $x \in [x_1, x_2]$.
So, we need 2 integrals for the Washer Method.

Sidenote. We also prefer the Cylindrical Shells Method here since if we use the Washer Method, the inner and outer radius of the washers for $x \in [x_0, x_1]$ are given by the same curve.

3 Let R be the region bounded by the curves $4y = x^2$ and $x = 2y - 4$.



To determine the intersections between $4y=x^2$, $y=\frac{1}{4}x^2$ and $x=2y-4$, $y=\frac{1}{2}(x+4)$:

$\frac{1}{4}x^2=\frac{1}{2}(x+4)$; $x^2=2(x+4)$; $x^2-2x-4=(x-4)(x+2)=0$;

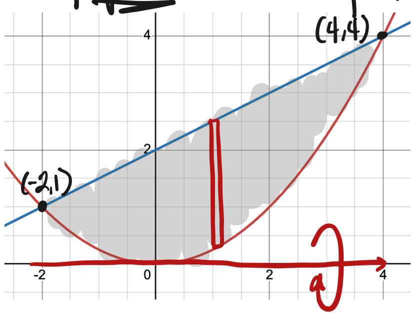
$x=-2,4$;

For $x=-2$: $y=\frac{1}{2}(-2+4)=1$; For $x=4$: $y=\frac{1}{2}(4+4)=4$;

Intersections @ $(-2,1)$ and $(4,4)$.

Part a: Washer method. Determine the volume obtained by rotating R about the x-axis.

Slice perpendicular to the axis of rotation.



thickness: dx

bounds: $x \in [-2,4]$

outer radius: $r_{outer} = y$ with $x=2y-4$: $r_{outer} = y = \frac{1}{2}(x+4)$;

inner radius: $r_{inner} = y$ with $4y=x^2$: $r_{inner} = y = \frac{1}{4}x^2$;

$$V = \pi \int \left[\begin{pmatrix}\text{outer}\\ \text{radius}\end{pmatrix}^2 - \begin{pmatrix}\text{inner}\\ \text{radius}\end{pmatrix}^2 \right](\text{thickness})$$

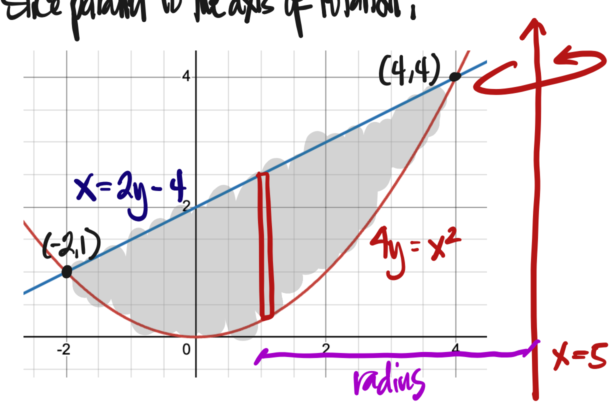
$$= \pi \int_{-2}^{4} \left[\tfrac{1}{2}(x+4)\right]^2 - \left[\tfrac{1}{4}x^2\right]^2 dx = \pi \int_{-2}^{4} \tfrac{1}{4}(x^2+8x+16) - \tfrac{1}{16}x^4\, dx$$

$$= \pi \int_{-2}^{4} \tfrac{1}{4}x^2 + 2x + 4 - \tfrac{1}{16}x^4\, dx = \pi \left[\tfrac{1}{4}\left(\tfrac{1}{3}\right)x^3 + 2\left(\tfrac{1}{2}\right)x^2 + 4x - \tfrac{1}{16}\left(\tfrac{1}{5}\right)x^5 \right]_{-2}^{4}$$

$$= \pi \left[\tfrac{1}{12}\left((4)^3-(-2)^3\right) + \left((4)^2-(-2)^2\right) + 4\left(4-(-2)\right) - \tfrac{1}{80}\left((4)^5-(-2)^5\right) \right] = \boxed{\tfrac{144}{5}\pi}$$

Part b: Shell Method. Determine the volume obtained by rotating R about the line $x = 5$.

Slice parallel to the axis of rotation.



thickness: dx

bounds: $x \in [-2,4]$

radius: $5-x$

height: $y_{high} - y_{low}$

with y_{high} : $x=2y-4$, $y_{high} = y = \frac{1}{2}(x+4)$

y_{low} : $4y=x^2$, $y_{low} = y = \frac{1}{4}(x^2)$

height $= \frac{1}{2}(x+4) - \frac{1}{4}(x^2) = \frac{1}{4}(2x+8-x^2)$

$$V = \int 2\pi(\text{radius})(\text{height})(\text{thickness}) = 2\pi \int_{-2}^{4} (5-x)\left(\tfrac{1}{4}\right)(-x^2+2x+8)\, dx \ ;$$

$$(-x+5)(-x^2+2x+8) = x^3 - 5x^2 - 2x^2 + 10x - 8x + 40 = x^3 - 7x^2 + 2x + 40 \ ;$$

$$V = \tfrac{1}{2}\pi \int_{-2}^{4} x^3 - 7x^2 + 2x + 40\, dx = \tfrac{1}{2}\pi \left[\tfrac{1}{4}x^4 - \tfrac{7}{3}x^3 + \cancel{2}\left(\tfrac{1}{\cancel{2}}\right)x^2 + 40x \right]_{-2}^{4}$$

$$= \tfrac{1}{2}\pi \left[\tfrac{1}{4}\left(4^4-(-2)^4\right) - \tfrac{7}{3}\left(4^3-(-2)^3\right) + \left(4^2-(-2)^2\right) + 40\left(4-(-2)\right) \right] = \tfrac{1}{2}\pi(144) = \boxed{72\pi}$$

Intersection bet. $y=-x$, $x=-y$ and $x=y^2-4y$:
$y^2-4y=-y$; $y^2+y-4y=y^2-3y=y(y-3)=0$, $y=0,3$

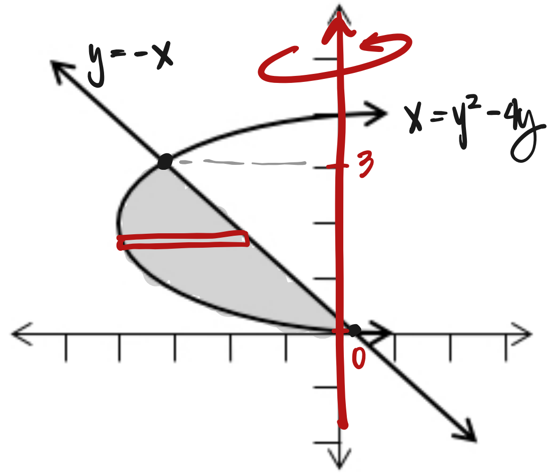
4 Let R be the region bounded by the curves $y=-x$ and $x=y^2-4y$.

a) Set up, but **do NOT evaluate**, the integral needed to determine the volume obtained by rotating R about the y-axis using the disc/washer method.

Slice perpendicular to the axis of rotation:

$$V=\pi\int\left[\left(\begin{smallmatrix}\text{outer}\\\text{radius}\end{smallmatrix}\right)^2-\left(\begin{smallmatrix}\text{inner}\\\text{radius}\end{smallmatrix}\right)^2\right](\text{thickness})$$

thickness: dy
bounds: $y\in[0,3]$
outer radius: $r_{outer}=x$ with $x=y^2-4y$; $r_{outer}=y^2-4y$
inner radius: $r_{inner}=x$ with $y=-x$; $r_{inner}=-y$



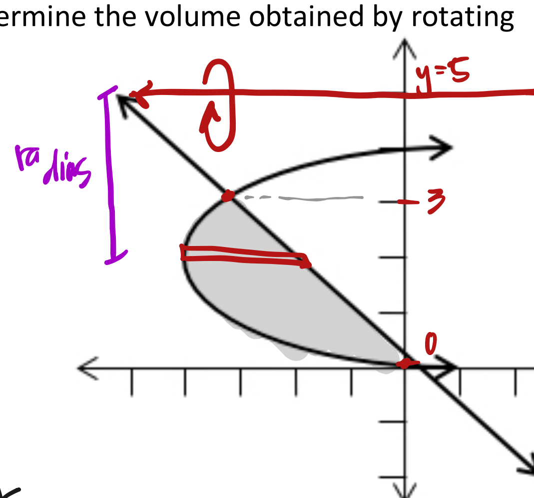
Integral:
$$V=\pi\int_0^3 (y^2-4y)^2-(-y)^2\,dy$$

b) Set up, but **do NOT evaluate**, the integral needed to determine the volume obtained by rotating R about the line $y=5$ using the shell method.

Slice parallel to the axis of rotation.

$$V=2\pi\int(\text{radius})(\text{height})(\text{thickness})$$

thickness: dy
bounds: $y\in[0,3]$
radius: $5-y$
height: $h_{up}=x$ with $y=-x$; $h_{up}=-y$
$h_{low}=x$ with $x=y^2-4y$; $h_{low}=y^2-4y$



Integral:
$$V=2\pi\int_0^3 (5-y)\left[(-y)-(y^2-4y)\right]dy$$

5. Use the general (cross-sectional) slicing method to find the volume of the following solid.

a) The solid whose base is the region bounded by the curves $y = x^2$ and $y = 2 - x^2$, and whose cross-sections through the solid perpendicular to the x-axis are semi-circles.

$V = \int \left(\text{cross-sectional area}\right)(\text{thickness})$

thickness: dx

bounds: $x \in [-1, 1]$

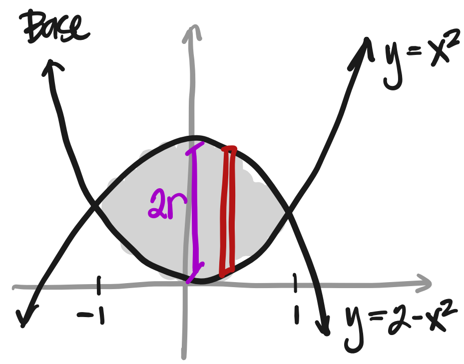
Intersection bet. $y = x^2$ and $y = 2 - x^2$

$x^2 = 2 - x^2$; $2x^2 = 2$; $x^2 = 1$; $x = \pm 1$.

area: $A = \frac{1}{2}\pi r^2$ with $r = \frac{1}{2}(y_{high} - y_{low})$

$y_{high}: y = 2 - x^2$, $y_{low}: y = x^2$

$A = \frac{1}{2}\pi(1 - x^2)^2 = \frac{1}{2}\pi(x^4 - 2x^2 + 1)$;



$$V = \int_{-1}^{1} \frac{1}{2}\pi(x^4 - 2x^2 + 1)\,dx = \frac{1}{2}\pi\left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^{1}$$

$$= \frac{1}{2}\pi\left[\frac{1}{5}(1^5 - (-1)^5) - \frac{2}{3}(1^3 - (-1)^3) + (1 - (-1))\right] = \frac{8}{15}\pi$$

V= $\boxed{\frac{8}{15}\pi}$

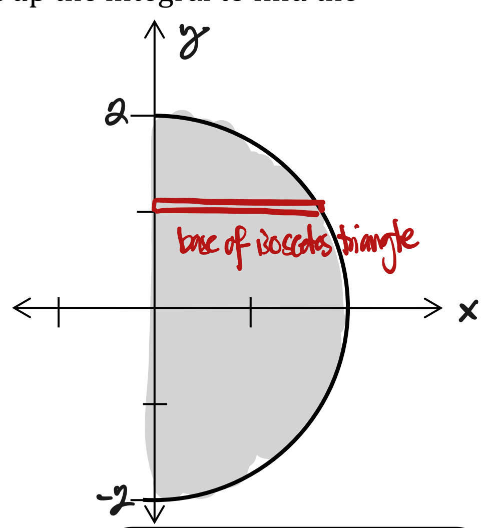
b) The base of a solid is the semi-circular region bounded by the curves $x = \sqrt{4 - y^2}$ and $x = 0$. Each cross-section perpendicular to the y-axis is a right-isosceles triangle with the right angle leg lying parallel to the x-axis (perpendicular to the y-axis). Set up the integral to find the volume of this solid.

thickness: dy

bounds: $y \in [-2, 2]$

area: $A = \frac{1}{2}bh = \frac{1}{2}b^2$ with $b = x$, $x = \sqrt{4 - y^2}$

$A = \frac{1}{2}(\sqrt{4 - y^2})^2 = \frac{1}{2}(4 - y^2)$



$$V = \int_{-2}^{2} A\,dy = \int_{-2}^{2} \frac{1}{2}(4 - y^2)\,dy$$

$$\overset{\text{symmetry}}{=} \cancel{2}\int_0^2 \cancel{\tfrac{1}{2}}(4 - y^2)\,dy = \left[4y - \frac{1}{3}y^3\right]_0^2$$

$$= 4(2 - 0) - \frac{1}{3}(2^3 - 0^3) = 8 - \frac{8}{3} = \frac{16}{3}$$;

V = $\boxed{\frac{16}{3}}$

3D View of Slice: